

# Superluminal propagation of solitary kinklike waves in amplifying media

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It is shown that solitary-wave, kinklike structures can propagate superluminally in two- and four-level amplifying media with strongly damped oscillations of coherences. This is done by solving analytically the Maxwell-Bloch equations in the kinetic limit. It is also shown that the true wave fronts—unlike the pseudo wave fronts of the kinks—must propagate with velocity  $c$ , so that no violation of special relativity is possible. The conditions of experimental verification are discussed.

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## I. INTRODUCTION

The propagation of optical signals in resonant linear and nonlinear media has been one of the most important research topics in quantum optics since the very beginning of that branch of physics. Among the landmark discoveries in the subject which have shaped our understanding of the resonant interaction of radiation with atoms one should mention, among other things, self-induced and electromagnetically induced transparency, optical solitons, self-focusing, superradiance, and propagation of photon echoes; see, e.g., [1–5]. Interest in the physics of pulse propagation has never ceased. However, a new stimulus to its further development has recently come from spectacular experiments demonstrating propagation with superluminal group velocities and without visible pulse deformation [6,7] as well as from experiments with slowing down, stopping, and reversing light pulses [8–14]. The theory of superluminal propagation was actively developed as early as the 1990s, and work on it is continued in the new century; see, e.g., [15–20]. In addition, superluminal propagation of wavelike structures was considered within the framework of the classical field theory and completely integrable systems [21,22]. In particular, the existence of very interesting solutions to a model completely integrable Lorentz-invariant system has been reported in [21]; those solutions describe the decay of a subluminal propagating kink (“bradyon”) with the emergence of another bradyon accompanied by a superluminally propagating kinklike wave which may be called a “tachyon.” Alongside theoretical developments, experimental demonstrations of superluminal effects have in the meantime become almost routine in increasingly simple systems, as shown, e.g., in [23,24].

Let us notice here that, to our knowledge, the majority of studies concerning the superluminal propagation of *electromagnetic* radiation are related to the propagation of Gaussian pulses or other pulses with well-defined group velocities. But if one insists that “superluminality” is an effect which should strongly resemble the breaking down of causality, pulses of different type should be examined. Indeed, in vacuum, it is

not the group velocity that is in any sense important from the point of view of special relativity. What is important is the traveling discontinuity—that is, the wave front. It is the propagation of the wave front of the electromagnetic field that is the key for Einstein’s gedanken experiments with synchronization and actually for the very definition of causality. That wave front always propagates with velocity  $c$ .

The purpose of this work is to study the propagation of solitary waves having the shape of kinks. Kinks are solitary waves which can be characterized by a very-well-defined front (called a pseudo wave front here) which, when it is sufficiently steep, resembles discontinuity. Therefore, if we find a kinklike structure with the superluminally propagating pseudo wave front, the illusion of causality breaking would reach its limitations. In that sense, we believe, experiments with faster-than- $c$  propagating kinks would be a maximum that can be achieved at all from the point of view of “superluminality.”

The kinks to be described below propagate in a homogeneously broadened medium in which oscillations of the atomic dipole moment are strongly damped, but in which the decay of the population of the excited atomic state is slow. We provide here an analytical discussion of the propagation via an approximate solution to the Maxwell-Bloch equations in their kinetic limit. We show that, under very reasonable approximations, the system allows for the propagation of kinks. If the medium is amplifying—i.e., if it is prepared with larger-than-zero inversion—the kinks propagate with superluminal velocity of their pseudo wave front. After some discussion of analytical and numerical results, we explain why the superluminally propagating kink does actually not violate causality or special relativity.

The propagation of signals in an atomic medium is usually described within the slowly varying envelope approximation (SVEA), which is also adopted here. This means, in particular, that we are not able to describe the Sommerfeld forerunners, which are always present in any dispersive medium, linear or not, and which propagate always with the velocity equal to  $c$ . The kinks that appear below are the shapes of the envelope of the signal, and they are very steep on temporal scales several orders larger than that associated with the frequency of the carrier wave or that of the atomic transitions. This point will be discussed below in some detail.

The main part of this work is organized as follows. In Sec. II we define the mathematical model of the two-level me-

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dium in which the kinks propagate and provide some analytical solutions with their discussion. Section III is devoted to the presentation of related numerical results and the demonstration that no causality breaking can actually happen. In Sec. IV we discuss a somewhat more complicated model in which the “doubly superluminal” propagation of the kink-antikink pairs can appear. Final remarks are contained in Sec. V.

## II. PROPAGATION IN TWO-LEVEL MEDIA: ANALYTICAL RESULTS IN THE KINETIC LIMIT

We consider the propagation of electromagnetic radiation in a medium which consists of two-level atoms. For our purposes it will be sufficient to restrict ourselves to one-dimensional propagation and to a single (TE) polarization of the linearly polarized field. We assume that the medium is such that the transverse decay rate (i.e., the decay rate of oscillations of the atomic dipole moment) is much larger than the coupling constant between the field and medium (as expressed in units of frequency). On the other hand, the coupling constant is much larger than the longitudinal decay rate (i.e., the decay rate of the population inversion in the atom). More precisely, it is assumed that the following chain of inequalities holds true:

$$\omega, \omega_0 \gg \gamma_T \gg \Omega(x, t), \nu \gg \gamma_L, \quad (1)$$

where  $\omega$  is the frequency of the carrier wave,  $\hbar\omega_0$  is the energy gap between the two atomic levels,  $\gamma_T$  is the transverse decay rate,  $\Omega(x, t) = 2\mathbf{d}_{12}\mathbf{F}(x, t)/\hbar$  is the Rabi frequency associated with the propagating electric field, the frequency  $\nu$  is given by

$$\nu = 2 \frac{N|\mathbf{d}_{12}|^2\omega}{\hbar\epsilon_0\gamma_T},$$

and  $\gamma_L$  is the longitudinal decay rate.

Since we restrict ourselves to the one-dimensional propagation and a single polarization of the electromagnetic field, we shall henceforth neglect the vectorial nature of the transition dipole moment  $\mathbf{d}_{12}$  and of the envelope  $\mathbf{F}$ , and will simply write  $d_{12}$  and  $F$ .

The inhomogeneous broadening will be neglected here, but it can and should be taken into account; we plan to do this on another occasion.

The semiclassical Maxwell-Bloch equations which describe the dynamics of the system in one spatial dimension within SVEA read [2]

$$\frac{\partial s_{22}}{\partial t} = i(s_{21} - s_{12}) \frac{d_{12}F}{\hbar} - \gamma_L s_{22}, \quad (2)$$

$$\frac{\partial s_{11}}{\partial t} = -i(s_{21} - s_{12}) \frac{d_{12}F}{\hbar} + \gamma_L s_{22}, \quad (3)$$

$$\frac{\partial s_{12}}{\partial t} = -i\Delta s_{12} - i(s_{22} - s_{11}) \frac{d_{12}F}{\hbar} - \gamma_T s_{12}, \quad (4)$$

$$\frac{\partial s_{21}}{\partial t} = i\Delta s_{21} + i(s_{22} - s_{11}) \frac{d_{12}F}{\hbar} - \gamma_T s_{21}, \quad (5)$$

$$\frac{\partial F}{\partial t} + \frac{kc^2}{\omega} \frac{\partial F}{\partial x} = -i \frac{Nd_{12}\omega}{2\epsilon_0} (s_{21} - s_{12}), \quad (6)$$

$$(\omega^2 - k^2c^2)F = -\frac{Nd_{12}\omega^2}{2\epsilon_0} (s_{21} + s_{12}), \quad (7)$$

where  $s_{ij}$  are slowly varying envelopes of the expectation values of the atomic lowering and raising operators (for  $i \neq j$ ) as well as populations (for  $i = j$ ),  $k$  is the wave number of the carrier wave,  $N$  is the number density of atoms, and  $\Delta$  is the detuning of the carrier wave from exact resonance with the atom.

From now on we consider only the case of exact resonance, so that  $\Delta = 0$  and  $\omega = kc$ . We perform now the adiabatic elimination of  $s_{12}$  and  $s_{21}$  (cf. [2]), which is possible because of our assumption that  $\gamma_T$  is the largest available quantity of the dimension of frequency except of  $\omega$ . In particular,  $\gamma_T$  is much larger than the Rabi frequency associated with the propagating field  $F$ .

After the free oscillations of  $s_{12}$  and  $s_{21}$  are damped, we may write

$$s_{12} = -i \frac{d_{12}}{\hbar\gamma_T} (s_{22} - s_{11})F, \quad (8)$$

$$s_{21} = i \frac{d_{12}}{\hbar\gamma_T} (s_{22} - s_{11})F, \quad (9)$$

so that

$$\frac{\partial s_z}{\partial t} = \frac{\partial (s_{22} - s_{11})}{\partial t} = -\frac{4d_{12}^2}{\hbar^2\gamma_T} s_z F^2 - \gamma_L s_z - \gamma_L. \quad (10)$$

For the slowly varying envelope of the electric field we obtain

$$\frac{\partial F}{\partial t} + c \frac{\partial F}{\partial x} = \frac{1}{2} \nu s_z F. \quad (11)$$

In the above equations  $s_z$  is the population inversion.

It is a bit more convenient to work with the intensity, rather than with the envelope  $F$ , and to employ dimensionless quantities. Let the dimensionless time  $\tau$  be, by definition, equal to  $\nu t$  and the dimensionless spatial variable  $\xi$  be equal to  $\nu x/c$ . In terms of  $\tau$  and  $\xi$  the velocity  $c$  is equal to 1. Equations (10) and (11) may be rewritten as

$$\frac{\partial s_z}{\partial \tau} = -s_z J - \epsilon s_z - \epsilon, \quad (12)$$

$$\frac{\partial J}{\partial \tau} + \frac{\partial J}{\partial \xi} = s_z J, \quad (13)$$

where

$$J = \frac{2d_{12}^2 F^2}{\hbar^2 \nu \gamma_T},$$

and where  $\epsilon = \gamma_L/\nu$ . If  $\gamma_L$  is so small that  $\epsilon$  can be neglected, we obtain, more trivially

$$\frac{\partial s_z}{\partial \tau} = -s_z J. \quad (14)$$

Our strategy is to gain intuition by solving Eqs. (13) and (14) exactly, and then take into account the presence of  $\epsilon$  via the numerical solution of Eqs. (12) and (13).

Inspired by the truncated Painleve expansion method, we have found the following general solution of Eqs. (13) and (14) [26]:

$$s_z(\tau, \xi) = -\frac{\psi'(\xi)}{\psi(\xi) + \chi(\tau - \xi)}, \quad (15)$$

$$J(\tau, \xi) = \frac{\chi'(\tau - \xi)}{\psi(\xi) + \chi(\tau - \xi)}, \quad (16)$$

where  $\psi(\xi)$  and  $\chi(\tau - \xi)$  are arbitrary functions of their arguments and a prime denotes differentiation over the argument of the respective function. The natural boundary conditions to be imposed are such that, for  $\tau \rightarrow -\infty$ ,  $J$  approaches zero, while  $s_z$  becomes a constant; the latter will be called  $w_0$ . If the system has not been prepared in a special way, one has  $w_0 = -1$ . Let us choose the function  $\chi$  to be the simple exponent

$$\chi(\tau - \xi) = A \exp[a(\tau - \xi)],$$

where  $a$  and  $A$  are positive constants. Then the solutions which satisfy the above boundary conditions are given by

$$s_z = \frac{w_0 \psi_0 \exp(-w_0 \xi)}{\psi_0 \exp(-w_0 \xi) + A \exp[a(\tau - \xi)]}, \quad (17)$$

$$J = \frac{aA \exp[a(\tau - \xi)]}{\psi_0 \exp(-w_0 \xi) + A \exp[a(\tau - \xi)]}, \quad (18)$$

and have the form of kinks. The most natural definition of the velocity of a kink is that it is equal to the velocity of the region of the fastest change of the dependent variable. To make this definition quantitative, let us define the velocity of the kink as the velocity of its inflection point, which is at the ‘‘center’’ of the kink, at the half of its height. Now, for the kinks in the solutions (17) and (18), the velocity of the inflection point is easy to obtain by calculating second derivatives over  $\xi$ . This way we obtain the velocity

$$v = \frac{a}{a - w_0}, \quad (19)$$

which means that for *positive*  $w_0$  the kink propagates superluminally and its velocity grows with growing inversion  $w_0$ .

A simple interpretation of the solutions (17) and (18) follows directly from the intuitive meaning of the equations of motion (13) and (14). The pulse of the light in the medium does not so much propagate but appears to be ‘‘built’’ from the positive inversion. The rate of change of the inversion is proportional to the intensity. When the intensity grows slowly, the inversion diminishes slowly as well, but with increasing intensity the decrease of inversion becomes very fast, to slow down again when the inversion itself is close to zero. In this way kinks are formed. The inversion and inten-

sity conspire to preserve the shapes of each other.

As mentioned before, the superluminal solutions shown above are particularly attractive because the kinks possess a very well-defined pseudo wave front resembling discontinuity. Thus, the propagation of a kink having the inflection point moving with velocity larger than  $c$  seems to be rather spectacular, making the illusion of superluminality almost true, especially that the shape of the signal is preserved.

Before proceeding further to somewhat more realistic kinks to be dealt with numerically, let us show that the analytical solutions do not break fundamental assumptions of the SVEA. On returning to the original variables  $x$  and  $t$ , and calculating the derivative, we find the ratios

$$\frac{|\partial F / \partial x|}{(\omega/c)|F|} = \frac{1}{2} |w_0 - a| \frac{\nu}{\omega} \frac{|\psi_0 e^{-w_0 \nu x/c}|}{|\psi_0 e^{-w_0 \nu x/c} + A e^{a \nu (t-x/c)}|} \leq \frac{1}{2} |a - w_0| \frac{\nu}{\omega}, \quad (20)$$

$$\frac{|\partial F / \partial t|}{\omega|F|} = \frac{1}{2} |a| \frac{\nu}{\omega} \frac{|\psi_0 e^{-w_0 \nu x/c}|}{|\psi_0 e^{-w_0 \nu x/c} + A e^{a \nu (t-x/c)}|} \leq \frac{1}{2} |a| \frac{\nu}{\omega}. \quad (21)$$

Thus, we can see that if the constant  $a$  is of the order of 1, the above ratios are very small indeed because of our assumptions about  $\nu$  and  $\omega$ . Similar relations hold for the ratios of second to first derivatives:

$$\frac{|\partial^2 F / \partial x^2|}{(\omega/c)|\partial F / \partial x|} \leq \frac{5}{2} |a - w_0| \frac{\nu}{\omega}, \quad (22)$$

$$\frac{|\partial^2 F / \partial t^2|}{\omega|\partial F / \partial t|} \leq \frac{5}{2} |a| \frac{\nu}{\omega}. \quad (23)$$

The important point here is that the pseudo wave fronts of the kinks are very steep on the time scales related to  $\nu$ , but on the time scales associated with the period of oscillations of the atomic dipoles the kinks can look completely flat. The increase in the amplitudes from zero to their maximal values happens in nanoseconds, not in femtoseconds.

### III. NUMERICAL RESULTS FOR THE PROPAGATION OF KINKS IN TWO-LEVEL MEDIA AND THE DYNAMICS NEAR THE WAVE FRONT

Our tasks are now, first, to prove that the presence of nonzero  $\gamma_L$  does not spoil the superluminal kink propagation; second, to sketch the idea of the appropriate experiment; and finally, to show explicitly that the *true* wave front propagates with the velocity  $c$ , as it must.

To deal with the first problem, we have solved numerically Eqs. (12) and (13) with the help of a homemade program based on the split-operator algorithm. The algorithm has been proved to be stable, and the results have been checked with the help of the method of lines based on the fourth-order Runge-Kutta solver. The results for the propagation of the signal, which at  $\xi=0$  has the form

$$J_{exc}(\tau) = \frac{J_0}{1 + e^{-a(\tau-20)}}, \quad (24)$$

with  $a=1$  and  $J_0=1$ , are shown in Fig. 1(a).

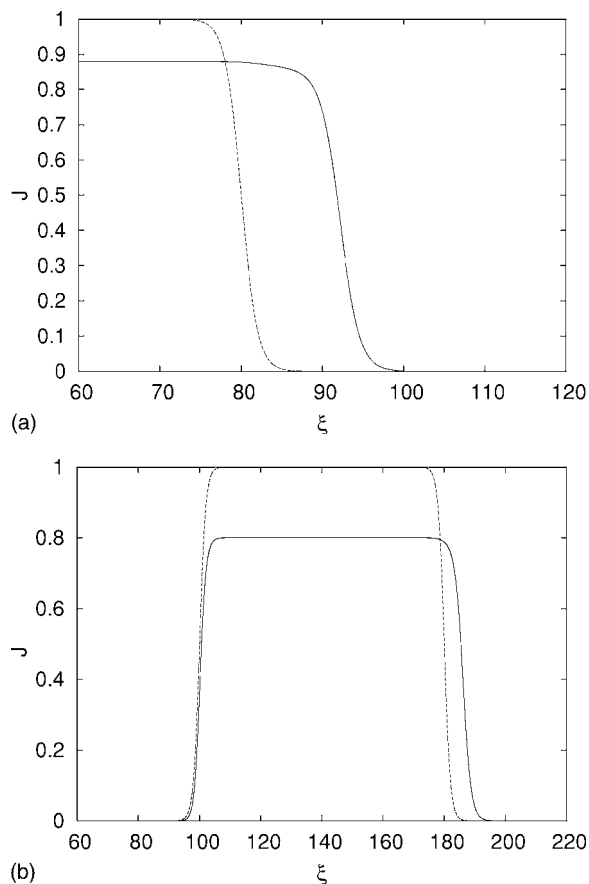


FIG. 1. Snapshots of (a) kink and (b) kink-antikink pair compared with the same objects which would have propagated at the same time in vacuum. In (a), the kink propagated through the two-level medium with the initial inversion  $w_0=0.5$ , occupying the spatial region from  $\xi=20$  to  $\xi=60$ . The longitudinal decay rate  $\gamma_L$  was equal to  $3 \times 10^{-3}\nu$ , and the maximal amplitude of the intensity was equal to 1. The kink which propagated through the two-level medium (solid line) is displayed together with the same kink which would have propagated in vacuum. The snapshot was taken at  $\tau=100$ . In (b), the same comparison is displayed, but for a kink-antikink pair. The decay rate is  $\gamma_L=5 \times 10^{-3}$ , and the other parameters are the same as in (a). The snapshot was taken at  $\tau=200$ .

Inside the medium, the intensity was initially equal to zero, while  $w_0=1/2$ . The medium was not infinite, but it occupied the region  $20 < \xi < 60$ . Thus, Fig. 1(a) shows the situation at the time when the pseudo wave front had already passed through the medium. Figure 1(a) shows also a signal which would have propagated in the vacuum if it had the boundary shape given by Eq. (24). Such a signal would have had the form

$$J_{exc}(\tau, \xi) = \frac{J_0}{1 + e^{-a(\tau-20-\xi)}}.$$

Superluminal propagation of the pseudo wave front of the kinks is clearly visible, so that the presence of the longitudinal decay does not change the conclusions drawn from the simplified analytical model where the damping is absent.

What is more, we also considered the kink-antikink pair propagating under the same conditions. It is displayed in Fig. 1(b). It is clear that both pseudo wave fronts of the pair propagate superluminally in the same direction. Experimenting with other parameters of the system we observed that the longitudinal decay larger than  $\approx 6 \times 10^{-3}\nu$  precludes faster-than- $c$  propagation. This is quite obvious because the medium ceases to be amplifying for faster decay of the excited-state population. What is more, for  $J_0$  substantially different from 1 (as well as for  $w_0 \approx +1$ ) the kink (or the kink-antikink pair) becomes strongly deformed. The value  $J_0=1$  corresponds to the maximal Rabi frequency of the applied field  $\Omega_{max}$  equal to the geometric mean of  $\gamma_T$  and  $\nu$ ,  $\Omega_{max} = \sqrt{\gamma_T \nu}$ . Thus, it seems the values of parameters chosen when plotting Fig. 1 are close to the optimal ones. This leads us to the consideration of the problem of realizing our superluminal kinks in the laboratory. First, we believe that obtaining inverted populations in a sample of two-level atoms (for the purpose of getting  $w_0 > 0$ ) has been rather routinely performed in recent experiments. For instance, our excited state can be thought of as a middle (lower excited) level in a pumped three-level system. The pump field couples the ground state with the higher excited state. It is assumed that the population of the latter state incoherently decays very fast to populate the lower excited state, while the pump field does not couple it with the ground state. Also, the rate of decay from the middle state to the ground state (this is our  $\gamma_L$ ) is very small. When the population of the middle level is sufficiently large, the probe (kinklike) excitation comes to couple the middle state with the ground state. To maintain positive inversion in the sample, the pump field can be kept turned on, but the inversion should not be exceedingly large. To control the velocity of the kink in the medium as given by Eq. (19), we have to be able to control the amplitude and steepness of the applied signal.

The inequalities to be satisfied by the parameters of the system lead to the following numbers: for optical frequencies of the applied field of the order of  $10^{15}$  Hz, we should have  $\gamma_L \sim 2.5 \times 10^7$  Hz,  $\nu \sim 5 \times 10^9$  Hz,  $\Omega_{max} \sim 5 \times 10^{10}$  Hz, and  $\gamma_T \sim 5 \times 10^{11}$  Hz, so that  $\Omega_{max} \approx \sqrt{\gamma_T \nu}$ . To check whether we can obtain superluminal kink propagation when those strict conditions for the parameters are not fulfilled, we have also performed a numerical analysis of the full Maxwell-Bloch equations within the SVEA. Its results are displayed in Fig. 2. Note that, while the unit of time in Fig. 1 is  $1/\nu$ , it is rather  $1/\gamma_T$  in Fig. 2. The parameters of the system used to plot Fig. 2 were the following:  $\Omega_{max}=0.22\gamma_T$ ,  $\nu=0.05\gamma_T$ , and  $\gamma_L=3 \times 10^{-4}\gamma_T$ . Thus, both  $\nu$  and  $\Omega_{max}$  were quite large when compared with  $\gamma_T$ , so that the system was not exactly within the regime of the kinetic limit. And yet, the superluminal propagation of the pseudo wave front of the kink as well of the kink-antikink pair is clearly visible when compared with analogous objects which would propagate in vacuum (drawn with a dashed line) while satisfying the same boundary conditions.

We believe that the relative stability of the kink-antikink pair observed here can be attributed to two facts. First, the time of the simulation is short. Second, the spatial region occupied by the amplifying medium is finite. The important factors here are (i) the time spent by the pair inside the am-

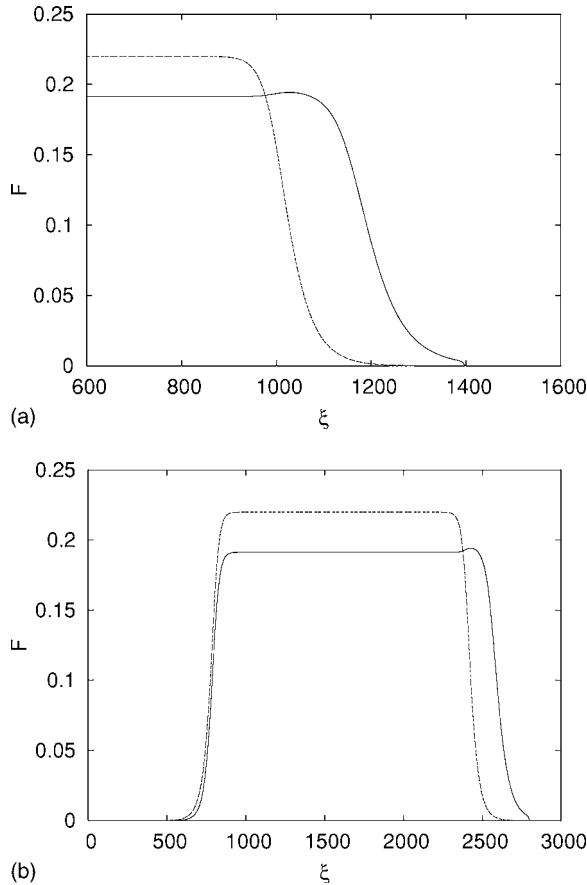


FIG. 2. Snapshots of the kinklike structures obtained from numerical solutions of the Maxwell-Bloch equations within the SVEA but without taking the kinetic limit. Snapshots of (a) kink and (b) kink-antikink pair which propagated in the two-level medium are compared with an analogous kink-antikink pair which would have propagated at the same time in vacuum. The boundary conditions at  $\tau=0$  are the same. The unit of time was equal to  $1/\gamma_T$ , the unit of space  $c/\gamma_T$ . In (a), the kink has propagated through the two-level medium with initial inversion  $w_0=0.5$ , occupying the spatial region from  $\xi=400$  to  $\xi=600$ . The longitudinal decay rate  $\gamma_L$  was equal to  $3 \times 10^{-4}\gamma_T$ , the maximal Rabi frequency of the electric field was equal to  $0.22\gamma_T$ , and the frequency  $\nu=0.05\gamma_T$ . The kink which propagated through the two-level medium (solid line) is displayed together with the same kink which would have propagated in vacuum. The snapshot was taken at  $\tau=1400$ . In (b), the same comparison is displayed, but for a kink-antikink pair, and the other parameters are the same as in (a). The snapshot was taken at  $\tau=2800$ .

plifying medium and (ii) the initial value of the inversion  $w_0$ . We have observed in our simulations that if the layer of the atomic medium is about 4 times thicker than in Fig. 1(b) (so that the kink-antikink pair remains inside the medium for about 4 times longer)—and all other parameters are kept unchanged—the pair becomes very strongly deformed. This happens through the strong damping of the kink part of the pair because the medium is no longer amplifying for large times since the inversion becomes lower than zero. On the other hand, if the initial inversion is close to 1, a very high peak close to the “true” wave front is formed, completely breaking the symmetry between the kink and antikink parts

of the pair. In any case, the pair eventually becomes disrupted. The values of parameters with the help of which our figures were plotted had resulted from an extensive numerical experimentation, guided, on the one hand, by the issue of the pair stability and, on the other hand, by the need to keep them experimentally accessible. Indeed, let us stress here that the spatial size of the medium in Fig. 1, equal to  $\Delta\xi=40$  in dimensionless units, corresponds to  $\Delta x=240$  cm, which is certainly realizable in the laboratory.

Finally, let us consider the problem of whether the superluminal propagation of the pseudo wave front of the kink violates special relativity. To do this, let us study the structure of solutions of Eqs. (12) and (13) near the true wave front of the envelope. One has to distinguish here between (i) the wave front of the complete signal (given by solutions to the full Maxwell-Bloch equations without the SVEA), (ii) that of the slowly varying envelope (as obtained from the Maxwell-Bloch equations within the SVEA), and, finally, (iii) the wave front of the envelope in the kinetic limit. For our purposes, it is sufficient to investigate the last of the three wave fronts. The full analysis of the former ones will be given elsewhere [27]. Naturally, we should first determine precisely where the wave front is—i.e., what its velocity can be equal to. To determine this, we adopt the method due to Whitham [28] (Chap. 5.6) and expand  $s_z$  and  $J$  in powers of the variable  $\zeta=\xi-X(\tau)$  with time-dependent coefficients,

$$s_z(\xi, \tau) = \sum_{n=0} s_z^{(n)}(\tau) \zeta^n \quad (25)$$

and

$$J(\xi, \tau) = \sum_{n=0} J^{(n)}(\tau) \zeta^n, \quad (26)$$

near an exact space-independent solution of the system. Here  $X(\tau)$  is the position of the wave front to be determined. In the original Whitham’s treatment the zeroth-order term is simply a constant, but here we allowed it to be time dependent. On the right-hand side of the wave front,  $J$  should be zero since the signal has not yet arrived there, while  $s_z$  should evolve freely—that is,  $s_z=(w_0+1)\exp(-\epsilon\tau)-1$ . Let us notice that  $J=0$  and  $s_z=(w_0+1)\exp(-\epsilon\tau)-1$  are special exact solutions to Eqs. (12) and (13). Thus, we write

$$s_z^{(0)}(\tau) = (w_0 + 1)\exp(-\epsilon\tau) - 1, \quad (27)$$

$$J^{(0)}(\tau) = 0. \quad (28)$$

On substituting the expansions (25) and (26) for Eqs. (12) and (13), equating the expression standing on both sides at the same powers of  $\zeta$ , and using Eqs. (27) and (28), we obtain, in zeroth-order,

$$-s_z^{(1)}(\tau)\dot{X}(\tau) = 0,$$

$$-J^{(1)}(\tau)[\dot{X}(\tau) - 1] = 0,$$

where the overdot denotes differentiation over  $\tau$ . Thus, there are only two possibilities: either  $\dot{X}=0$  and  $J^{(1)}=0$  (with  $s_z^{(1)}$  being still undetermined) or  $\dot{X}=1$  and  $s_z^{(1)}=0$ , while  $J^{(1)}$  is to

be determined from a higher-order equation. The former possibility leads to the trivial solution with all  $J^{(n)}=0$  only (let us notice, though, that this is not true for the Maxwell-Bloch equations without kinetic limit). The remaining possibility is that  $\dot{X}=1$ , so that the velocity of the true wave front of the envelope in the kinetic limit is still equal to  $c$  in the standard units. For  $J^{(1)}(\tau)$  we obtain the following solution from the first-order terms in  $\zeta$ :

$$J^{(1)}(\tau) = J^{(1)}(0) \exp\left(\frac{w_0 + 1}{\epsilon}(1 - e^{-\epsilon\tau}) - \tau\right),$$

which means that the system is linearly unstable for  $w_0 > 0$ , but only as long as  $\tau$  is smaller than, approximately,  $1/\epsilon$ . In particular, the wave front of the envelope gets steeper, which is quite an intuitive result. Similarly, from the first-order terms in  $\zeta$  we obtain  $s_z^{(2)}$ :

$$s_z^{(2)} = C \exp\left(\frac{w_0 + 1}{\epsilon}(1 - e^{-\epsilon\tau}) - \tau - \epsilon\tau\right)(1 - e^{\epsilon\tau} + w_0),$$

where  $C$  is a constant. Thus, in addition, we can conclude that the behavior of the inversion is smoother than that of the intensity, since the former exhibits the discontinuity of the second derivative, while it is the first derivative of the intensity which is discontinuous at the wave front. But the crucial result of the above simple calculations is the velocity of the true wave front of the envelope, being equal to 1 or  $c$  in the standard units. Let us notice by the way that the fact that only the first derivative of  $J$ , and not  $J$  itself, is discontinuous should be attributed to the nonexistence of nonzero constant solutions for the intensity.

We have defined the true, ‘‘genuine’’ wave front as the surface in space-time on which either the field itself or any of its derivatives exhibit discontinuity. This notion is a generalization of the standard, intuitive concept of the wave front as the surface which divides the space into two regions: that where the field is already present and that which has not yet been reached. Following the classic arguments of Einstein’s theory of relativity, the existence of such surfaces of discontinuity is a condition of the very existence of the causality principle. On the other hand, along the slope of the kink there is no discontinuity of either the field itself or its derivatives: the slope may be very steep, but it is still smooth. As such, it cannot be called the true wave front. However, from the experimental point of view, registering the true wave front is extremely difficult. On the contrary, the pseudo wave front formed by the slope of the kink is detectable, and when observed in a laboratory, it would strongly resemble the discontinuity. This is the reason why the kinklike solutions are so interesting.

Let us finally notice that the near-the-wave-front analysis which we employed following Whitham holds for arbitrary signals and not just kinks. Thus, in our system we have to do with two frontlike solutions, one of them luminal and the second superluminal.

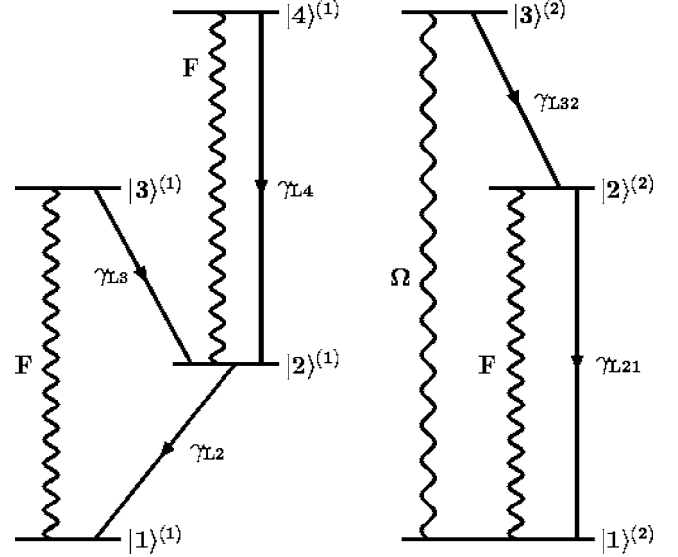


FIG. 3. Energy-level scheme of the two kinds of atoms which constitute the system. The superscripts (1) and (2) refer to those two atomic subsystems. The first subsystem contains four-level atoms coupled to the probe field in the  $\mathcal{N}$  configuration, and the longitudinal decay rates  $\gamma_{L3}$ ,  $\gamma_{L4}$  are supposed to be much larger than  $\gamma_{L2}$ . The system subsystem consists of three-level atoms coupled to both the pump field of the Rabi frequency  $\Omega$  and to the probe field having the slowly-varying amplitude  $F$ . It is assumed that  $\Omega$  is much larger than  $2d_{12}F/\hbar$  and that  $\gamma_{L32} \gg \gamma_{L21}$ , so that the inversion  $s_{22}^{(2)} - s_{11}^{(2)}$  is approximately constant, and the atoms of the second kind are coupled to the field approximately linearly.

#### IV. PROPAGATION OF KINKS IN FOUR-LEVEL MEDIA WITH REVERSE SATURATION

In the above example, we have shown that either the pseudo wave front of a kink or one of the pseudo wave fronts of the kink-antikink pair can propagate superluminally. However, one can find a related system in which both pseudo wave fronts of the kink-antikink pair propagate superluminally in *opposite* directions. Namely, let us consider a system of two kinds of atoms coupled to the electromagnetic field as shown in Fig. 3.

Atoms of the first kind are coupled to the field in the  $\mathcal{N}$  configurations with reversed saturation. It is assumed that the decay rates  $\gamma_{L4}$  and  $\gamma_{L3}$  are so large that the populations of the third and fourth levels are effectively zero, so that  $s_{11} + s_{22} \approx 1$ . What is more, it is assumed that the transverse decay rates are much larger than  $\gamma_{L2}$ . They should also be much larger than the Rabi frequency of the propagating field (at any spatial point). Under those conditions the nondiagonal expectation values of the atomic operators can be adiabatically eliminated.

The atoms of the second kind are coupled to the field in a  $V$  configurations, with the pump field having the Rabi frequency  $\Omega$ . It propagates perpendicularly to the direction of propagation of the probe field, the latter having the slowly varying amplitude  $F$ , and it is assumed that  $2d_{12}F/\hbar \ll \Omega$ . The pump field couples the first and third levels, and keeps the population of the second level almost constant since it is assumed that  $\gamma_{L32} \gg \gamma_{L21}$ . This subsystem is actually very

similar to the one considered previously, but now we must require that the pump field be turned on all the time. We also assume that the decay of oscillations of the dipole moment is much faster than that of the population of the second level. Under those conditions the system of three-level atoms works as a linear amplifying medium. Thus, the equations of motion for the population of the ground state of the four-level system and those for the dimensionless intensity  $L$  of the probe field read (cf. [25])

$$\frac{\partial s_{11}}{\partial \tau} = -s_{11}L + \Gamma_2(1 - s_{11}), \quad (29)$$

$$\frac{\partial L}{\partial \tau} + \frac{\partial L}{\partial \xi} = (\alpha - 1)s_{11}L + (\beta - \alpha)L, \quad (30)$$

where  $\tau = \nu t$  and  $\xi = \nu x/c$  are dimensionless temporal and spatial variables,  $\nu = 2N^{(1)}d_{13}^2\omega/(\hbar\epsilon_0\gamma_{T1}^{(1)})$ ,  $L$  is a dimensionless intensity,  $L = (2d_{13}F)^2/(2\hbar^2\gamma_{T1}^{(1)}\nu)$ , and  $\alpha$  is the ratio  $d_{24}^2\gamma_{T1}^{(1)}/(d_{13}^2\gamma_{T2}^{(1)})$ . The constants  $\gamma_{T1}^{(1)}$  and  $\gamma_{T2}^{(1)}$  are the decay rates of oscillations of  $s_{13}$  and  $s_{24}$ , respectively. The superscript  $(i)$  indicates that the parameters refer to the atoms of the  $i$ th kind,  $i=1,2$ . The dimensionless decay rate  $\Gamma_2$  is equal to  $\gamma_{L21}/\nu$ . The constant  $\beta$  depends, naturally, on the Rabi frequency  $\Omega$  of the pump field. Let us notice that the velocity  $c$  is again equal to 1 in terms of the above variables.

The above system of equations admits the following exact solution in terms of the kinklike solitary waves:

$$s_{11} = 1 - \frac{\beta - 1}{\alpha - 1} \frac{\phi_1 \exp[a(b\tau - \xi)]}{\phi_0 + \phi_1 \exp[a(b\tau - \xi)]}, \quad (31)$$

$$L = ab \frac{\phi_1 \exp[a(b\tau - \xi)]}{\phi_0 + \phi_1 \exp[a(b\tau - \xi)]}, \quad (32)$$

where  $a = (\beta - 1)/(b - 1)$  and

$$b = \frac{\Gamma_2}{\Gamma_2 + \beta - \alpha} \quad (33)$$

is the velocity of the pseudo wave front of the kink. There are three regimes in which the above solution is physically meaningful (which means that  $0 < s_{11} < 1$  and  $L > 0$ ): (i)  $\alpha < \beta < 1$  (then  $0 < b < 1$ ); (ii)  $\alpha > \beta > 1$ ,  $\Gamma_2 > \alpha - \beta$  (then  $b > 1$ ); and (iii)  $\alpha > \beta > 1$ ,  $\Gamma_2 < \alpha - \beta$  (then  $b < 0$ ). In the third regime of parameters we obtain the pseudo wave front of the kink propagating backwards—that is, in the direction opposite to that of the carrier wave. What is more, if the condition  $\alpha - 2\Gamma_2 < \beta < \alpha - \Gamma_2$  is fulfilled, we have  $b < -1$ ; i.e., the backward propagation of the pseudo wave front is superluminal. To check whether the above rather exotic behavior can be observed under more realistic conditions, we analyzed numerically the behavior of a kink-antikink pair. To integrate Eqs. (29) and (30) we again employed a homemade FORTRAN program based on a split-operator algorithm. The results were checked with the help of a method-of-line solver based on the fourth-order Runge-Kutta algorithm, using the routine ROCK4.F, freely available on the Internet. The initial conditions were taken as

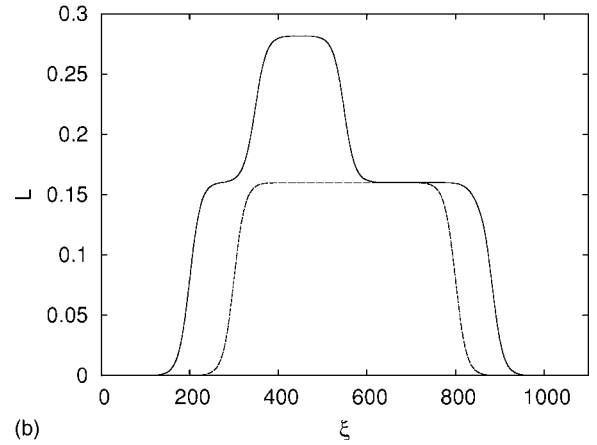
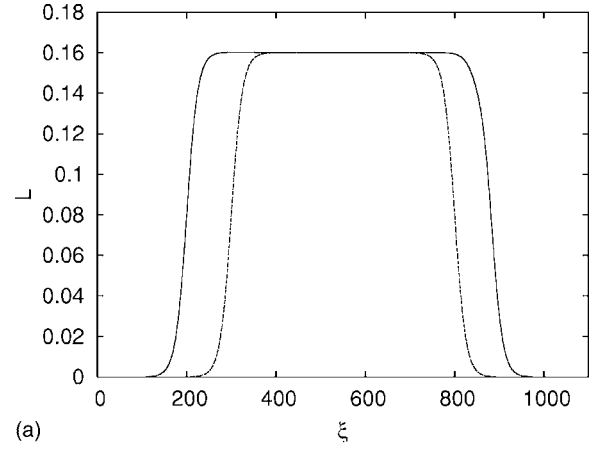


FIG. 4. Snapshots of the kink-antikink pair given by numerical solutions of the Maxwell-Bloch equations in the kinetic limit for the medium containing two kinds of atoms as specified in Sec. IV: (a) The kink-antikink pair at  $\tau=50$  (solid line) and at  $\tau=0$  (dashed line); at  $\tau=0$ , the central parts of the slope of the initial ground-state population matched the central part the antikink part of the initial intensity (see the text). (b) The same as in (a) but for the *mismatch* in the initial central parts of the population kink and the intensity antikink. Refer to the text for parameters. The time unit was equal to  $1/\nu$  in both (a) and (b).

$$s_{11}(\xi, 0) = 1 - \frac{\beta - 1}{\alpha - 1} \frac{\phi_1 \exp[a(\xi_1 - \xi)]}{\phi_0 + \phi_1 \exp[a(\xi_1 - \xi)]}, \quad (34)$$

$$L(\xi, 0) = ab\phi_1 \left( \frac{\exp[a(\xi_2 - \xi)]}{\phi_0 + \phi_1 \exp[a(\xi_2 - \xi)]} - \frac{\exp[a(\xi_3 - \xi)]}{\phi_0 + \phi_1 \exp[a(\xi_3 - \xi)]} \right), \quad (35)$$

with  $\phi_0 = \phi_1 = 1$ . The results are shown in Fig. 4, where the location of the kink-antikink pair was displayed for two times  $\tau=0$  and  $\tau=50$ . The parameters used to plot Fig. 4 were  $\alpha = 1.255$ ,  $\beta = 1.24$ , and  $\Gamma_2 = 0.01$ , so that  $b = -2$ . Part (a) of Fig. 4 illustrates the situation for  $\xi_1 = \xi_2 = 300$ ,  $\xi_3 = 800$ , so that the central part of the curve of the initial population matches the central part of the initial pseudo wave front of the antikink. On the other hand, part (b) was obtained when

there initially was a mismatch—that is, for  $\xi_1=500$  with other  $\xi$ 's as before. It is clear from that figure that the left pseudo wave front traveled a distance approximately equal to  $\Delta\xi=100$ , which agrees well with the theoretical predictions given by Eq. (33), while the right pseudo wave front also traveled superluminally with velocity approximately equal to 1.5. Let us notice that the kink-antikink pair remained almost undeformed in shape in (a), although the pulse became much broader. On the other hand, the mismatch in the initial conditions leads to a strong distortion of the signal [as seen in Fig. 4(b)], although our conclusions about the velocities of both pseudo wave fronts remain intact.

Experimental verification of the above predictions seems difficult. It is required that, when the kink-antikink pair enters the medium, the population of the ground state should be prepared according to Eq. (31). Ideally, the central part of the pseudo wave front of the kinklike population of the ground state at  $\tau=0$  would match the center of the pseudo wave front of the antikink part of the pair. What is more, if the amplitude of the pair significantly differs from that of Eq. (32), the pair becomes strongly deformed. Nevertheless, the feature of the superluminal propagation of both pseudo wave fronts is maintained. We believe that the predictions given above are so interesting that they would justify the effort of verification in the laboratory.

## V. FINAL REMARKS

To summarize, we have shown that a system of two-level atoms with inverted populations and strong damping of the dipole moment is a medium in which kinklike solitary waves can travel with superluminal velocity of the pseudo wave front. This should be particularly attractive for experimental verification because the pseudo wave front of the kink strongly resembles the genuine wave front of the electromagnetic field as it propagates in vacuum. Because of that, we believe that our proposal is, so to say, the “utmost” or “maximal” one. That is, in the context of matter-field interactions,

there are, basically, no other circumstances under which the illusion of causality breaking could come closer to reality. We have also discussed the requirements for the possible experimental realization of superluminal kinks. It has been shown numerically that those solitary waves can survive the presence of nonzero longitudinal damping. We have also found that the true wave front of the envelope propagates with the velocity  $c$ , as expected, so that there is no violation of Einstein's theory of relativity. The behavior of the intensity of the signal and the inversion near the true envelope wave front have been characterized quantitatively. What is more, we have found a system in which the pseudo wave front of the antikink structure can also propagate superluminally in a direction opposite to that of the carrier wave. This can happen, for instance, in a medium composed of (a) four-level atoms in the  $\mathcal{N}$  configuration and (b) three-level atoms under the presence of external pumping. Our predictions about superluminal backward propagation of the pseudo wave front of the antikink follow from an analytical solution confirmed by numerical calculations. The latter has also led us to the prediction of the superluminal propagation of both pseudo wave fronts of the kink-antikink pair. Thus, we can have to do with both the superluminal and reversed propagation of the same signal, and the reversed propagation is also faster than  $c$ .

Our analytical solutions have been shown, *a posteriori*, to satisfy conditions of validity of the main approximation employed, which is the slowly varying envelope approximation. Our numerical solutions also satisfy those conditions because of several temporal scales present in the system. In the time scale associated with a frequency  $\nu$ , the change of the intensity or inversion can be very fast near the pseudo wave front, but in the time scale associated with the frequency of the carrier wave the slope of pseudo wave fronts is very small.

## ACKNOWLEDGMENT

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